14.5. The chain rule

Recall : For a function $f(x)$ with x being a function of t, we have

$$\frac{df}{dt} = \frac{df}{dx} \cdot \frac{dx}{dt}$$

Prop For a function $f(x,y,z)$ with x,y,z being functions of t, we have

$$\frac{df}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial f}{\partial z} \cdot \frac{dz}{dt}$$

Note (1) This is related to the differential relation

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dt .$$

(2) If x,y,z are multi-variable functions, we use the partial derivatives $\frac{\partial x}{\partial t}, \frac{\partial y}{\partial t}, \frac{\partial z}{\partial t}$.

Thm (Implicit function theorem)

Given an equation $f(x,y,z) = 0$ with f differentiable,

$$\frac{\partial z}{\partial x} = -\frac{f_x}{f_z} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{f_y}{f_z}$$

Ex Given $f(x,y,z) = e^{x^3+yz^2}$ with $x = t^3 - 2t$, $y = t-1$, $z = t^2$,

find $\frac{df}{dt}$ for $t=1$.

Sol $$\frac{df}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial f}{\partial z} \cdot \frac{dz}{dt}$$

$$t=0 \Rightarrow x = 1^3 - 2\cdot 1 = -1,\quad y = 1-1 = 0,\quad z = 1^2 = 1.$$

$$\frac{\partial f}{\partial x} = e^{x^3+yz^2} \cdot 3x^2 = e^{-1^3+0\cdot 1^2} \cdot 3(-1)^2 = 3e^{-1}.$$

$$\frac{\partial f}{\partial y} = e^{x^3+yz^2} \cdot z^2 = e^{-1^3+0\cdot 1^2} \cdot 1^2 = e^{-1}.$$

$$\frac{\partial f}{\partial z} = e^{x^3+yz^2} \cdot 2yz = e^{-1^3+0\cdot 1^2} \cdot 2\cdot 0\cdot 1 = 0.$$

$$\frac{dx}{dt} = 3t^2 - 2 = 3\cdot 1^2 - 2 = 1.$$

$$\frac{dy}{dz} = 1,\quad \frac{dz}{dt} = 2t = 2.$$

$$\Rightarrow \frac{df}{dt} = 3e^{-1}\cdot 1 + e^{-1}\cdot 1 + 0\cdot 2 = \boxed{4e^{-1}}$$

Note You can directly derive $e^{x^3+yz^2} = e^{(t^3-2t)^3 + (t-1)t^2}$ with respect to t using the single-variable chain rule.

Ex Consider a function $g(x,y)$ with $x = st$, $y = s+t$.

(1) Compute $\frac{\partial g}{\partial s}$ and $\frac{\partial g}{\partial t}$.

Sol

$$\frac{\partial g}{\partial s} = \frac{\partial g}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial g}{\partial y}\cdot\frac{\partial y}{\partial s} = \boxed{g_x\cdot t + g_y\cdot 1}$$

chain rule

$$\frac{\partial g}{\partial t} = \frac{\partial g}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial g}{\partial y}\cdot\frac{\partial y}{\partial t} = \boxed{g_x\cdot s + g_y\cdot 1}$$

(2) Compute $\frac{\partial^2 g}{\partial s\partial t}$

Sol

$$\frac{\partial^2 g}{\partial s\partial t} = \frac{\partial}{\partial s}\left(\frac{\partial g}{\partial t}\right) = \frac{\partial}{\partial s}(g_x s + g_y)$$

$$= \frac{\partial g_x}{\partial s} s + g_x\cdot 1 + \frac{\partial g_y}{\partial s}$$

product rule

$$\frac{\partial g_x}{\partial s} = \frac{\partial g_x}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial g_x}{\partial y}\cdot\frac{\partial y}{\partial s} = g_{xx}\cdot t + g_{xy}\cdot 1$$

chain rule

$$\frac{\partial g_y}{\partial s} = \frac{\partial g_y}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial g_y}{\partial y}\cdot\frac{\partial y}{\partial s} = g_{yx}\cdot t + g_{yy}\cdot 1$$

$$\Rightarrow \frac{\partial^2 g}{\partial s\partial t} = (g_{xx}t + g_{xy})s + g_x + (g_{yx}t + g_{yy})$$

$$= \boxed{g_{xx}st + (s+t)g_{xy} + g_x + g_{yy}}.$$

Ex Given $e^z = xyz$, find $\frac{\partial z}{\partial x}$.

Sol 1 (Direction computation)

To find $\frac{\partial z}{\partial x}$, we derive the given equation with respect to x, regarding z as a function of x.

$$e^z = xyz \Rightarrow \frac{\partial}{\partial x}(e^z) = \frac{\partial}{\partial x}(xyz)$$

$$\frac{\partial}{\partial x}(e^z) = e^z \cdot \frac{\partial z}{\partial x}$$

$$\frac{\partial}{\partial x}(xyz) = \frac{\partial}{\partial x}(xy)\, z + xy\,\frac{\partial}{\partial x}(z) = yz + xy \cdot \frac{\partial z}{\partial x}$$

product rule

$$\Rightarrow e^z \cdot \frac{\partial z}{\partial x} = yz + xy \cdot \frac{\partial z}{\partial x} \Rightarrow (e^z - xy)\frac{\partial z}{\partial x} = yz$$

$$\Rightarrow \frac{\partial z}{\partial x} = \boxed{\frac{yz}{e^z - xy}}$$

Sol 2 (Implicit function theorem)

$$e^z = xyz \rightsquigarrow e^z - xyz = 0.$$

Set $f(x,y,z) = e^z - xyz$.

$$\Rightarrow \frac{\partial z}{\partial x} = -\frac{f_x}{f_z} = -\frac{-yz}{e^z - xy} = \boxed{\frac{yz}{e^z - xy}}$$